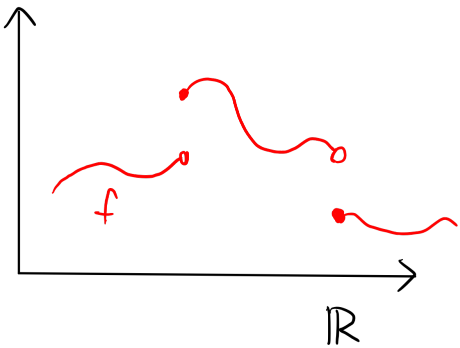
Analysis IV

(for EL, GM, MX)

Week 8 - 07.04.25

Review of Fourier transform

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a (piecewise) continuous function.



Fourier transform of f:

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} f(x)\, dx$$

Sometimes we also use the notation

$\mathcal{F}[f](a)$ for $\hat{f}(a)$.

In order for the integral to be well-defined:

We need $\int_{-\infty}^{+\infty} |f(x)|\, dx$ $\quad$ $(f \in L^1(\mathbb{R}))$

Inverse Fourier transform:

$$F^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{+iax} f(a)\, da$$

It is the inverse of F:

$$F^{-1}[F[f]] = f.$$

Note: The Fourier transform maps

$$\{f : \mathbb{R} \to \mathbb{C}\} \quad \text{to} \quad \{\hat{f} : \mathbb{R} \to \mathbb{C}\}.$$

In general: Even if f is real valued, $\hat{f}$ might be $\mathbb{C}$-valued.

(But, if f is real valued and even:
$\hat{f}$ is also real valued)

"Physical" domain $\quad f \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{f}\quad$: "Frequency" Domain

or time-domain

or space-domain

...

Properties of the Fourier transform:

1) Linearity: $\mathcal{F}[af+bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$

if $a, b \in \mathbb{R}$

2) Time shift: Let $g(x) = f(x-x_0)$, then

$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} f(x - x_0)\, dx$$

$$\overset{y = x - x_0}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ia(y + x_0)} f(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-iax_0} \int_{-\infty}^{+\infty} e^{-iay} f(y)\, dy$$

$$= e^{-iax_0} \hat{f}(a)$$

3) Frequency shift: Let $g(x) = e^{ia_0 x} f(x)$.

Then:

$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} \cdot e^{ia_0 x} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(a-a_0)x} f(x)\, dx$$

$$= \hat{f}(a-a_0)$$

Note: Property ③ is essentially like property ② but for the inverse Fourier transform. This is to be expected, since the inverse Fourier transform is defined with a similar expression as the Fourier transform (with e^{+iax} in place of e^{-iax})

4) Re-scaling (or dilation)

Let $g(x) = f(\lambda x)$ for some $\lambda \neq 0$.

Then:

$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} f(\lambda x)\, dx.$$

I want to substitute $y = \lambda x$. If $\lambda > 0$, then I get $\int_{-\infty}^{+\infty} \dots \frac{dy}{\lambda}$,

while if $\lambda < 0$ I get $\int_{+\infty}^{-\infty} \dots \frac{dy}{\lambda} = \int_{-\infty}^{+\infty} \dots \frac{dy}{|\lambda|}$

In both cases:

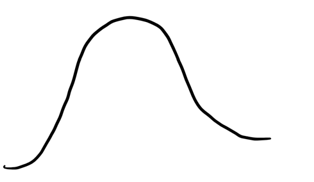
$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ia\frac{y}{\lambda}} \cdot f(y) \frac{dy}{|\lambda|}$$

$$= \frac{1}{|\lambda|} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\frac{a}{\lambda}y} f(y)\, dy$$

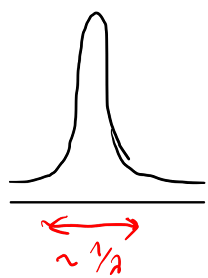
$$= \frac{1}{|\lambda|} \hat{f}\left(\frac{a}{\lambda}\right).$$

Remark: If the graphs of f and $\hat{f}$ look as follows:

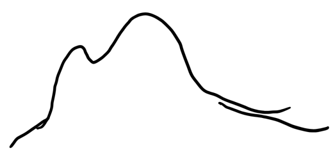
f:



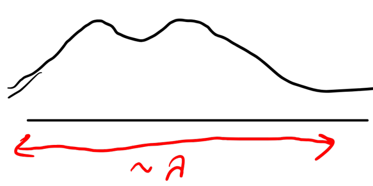
then, as $\lambda \to +\infty$, the graph of $f(\lambda x)$ is



$\hat{f}$:



and of $\frac{1}{|\lambda|} \hat{f}\left(\frac{a}{\lambda}\right)$:



So: When f is very concentrated around a point, $\mathcal{F}[f]$ is very spread out. This is a manifestation of the uncertainty principle.

5) Derivatives and Fourier transform:

Let $g(x) = f'(x)$.

Then $\hat{g}(a) = ia\,\hat{f}(a)$.

More generally:

$$\mathcal{F}[f^{(n)}](a) = (ia)^n\,\mathcal{F}[f](a)$$

So the Fourier transform: Transforms differential equations with constant coefficients to algebraic equations on the frequency domain.

Convolution: Given $f, g : \mathbb{R} \longrightarrow \mathbb{C}$, the convolution $f*g$ is defined by

$$f*g(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

Important properties:

i) Using the change of variables $\bar{y} = x - y$, one can see that:

$$\int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy = \int_{-\infty}^{+\infty} f(x-\bar{y})\, g(\bar{y})\, d\bar{y}.$$

So: $$f*g = g*f$$

ii) $$f*(g*h) = (f*g)*h$$

iii) $f*(g+h) = f*g + f*h$

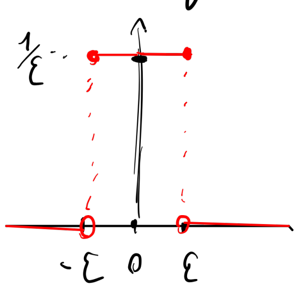
iv) $(f*g)' = f'*g = f*g'$

So $f*g$ is at least as smooth as the smoother among f, g.

One way to interprete $f*g$: As the "local average" of f with weights g.

Important example:

If $g(x) = \begin{cases} \frac{1}{2\varepsilon}, & x \in [-\varepsilon, \varepsilon], \\ 0, & \text{else}, \end{cases}$



Then:

$$f * g(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy =$$

$$= \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y)\, dy$$

So it's really the local average of f; Note that, in this case, as $\varepsilon \longrightarrow 0$, $f * g(x) \longrightarrow f(x)$.

The last point is true more generally:

If $g_\lambda(x)$ is a family of functions with $\int_{-\infty}^{+\infty} g_\lambda(x)\, dx = 1$ and $g_\lambda(x) = 0$ outside $[-\lambda, \lambda]$, then $f * g_\lambda(x) \xrightarrow{\lambda \to 0} f(x)$.

(In this case: As $\lambda \to 0$, $g_\lambda(x)$ looks like a δ-function).

6) Fourier transform and convolutions:

Let $f, g : \mathbb{R} \to \mathbb{C}$, then

$$\mathcal{F}[f * g](a) = \sqrt{2\pi} \cdot \mathcal{F}[f](a) \cdot \mathcal{F}[g](a).$$

So: $\mathcal{F}$ transforms convolutions into products (and the opposite as well, since $\mathcal{F}^{-1}$ has similar properties as $\mathcal{F}$).

In applications: We can often express

$\mathcal{F}[f]$ as the product of known functions, in which case we recover f as a convolution.

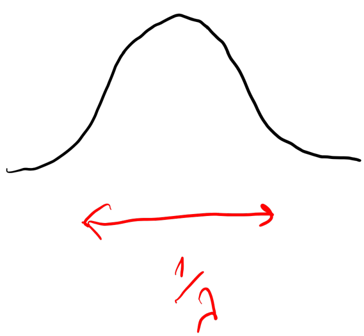
7) For the Gaussian function:

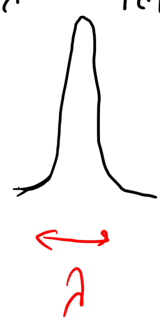
If $f(x) = e^{-x^2/2}$, $\hat{f}(a) = e^{-a^2/2}$

(We will prove it in the exercises).

So from property 4: For $\lambda \neq 0$,

If $f_\lambda(x) = e^{-\lambda^2 \frac{x^2}{2}} \Rightarrow \hat{f}_\lambda(a) = \frac{1}{|\lambda|} e^{-\frac{a^2}{2\lambda^2}}$





8) Plancherel's theorem:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}(a)|^2 da$$

(same energy for a signal and its Fourier transform).

How does complex analysis enter the study of the Fourier transform?

Let's see an example:

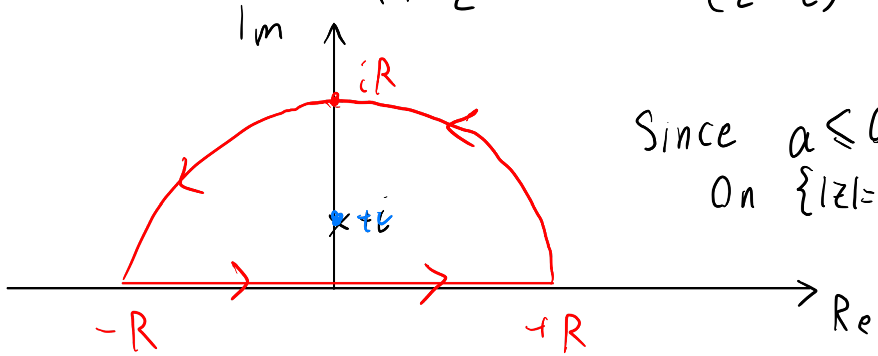
Let $f(x) = \frac{1}{1+x^2}$, then

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} \frac{1}{1+x^2} dx.$$

We can evaluate this integral as an application of the residue theorem:

Let $a \leqslant 0$ (similar analysis if $a > 0$).

If $f(z) = \dfrac{e^{-iaz}}{1+z^2} = \dfrac{e^{-iaz}}{(z-i)(z+i)}$



Since $a \leqslant 0$:
On $\{|z|=R\} \cap \{\mathrm{Im}(z) \geqslant 0\}$
f decays
as $R \to +\infty$

From the examples we have already calculated in class $\left(\text{For } \dfrac{P(x)}{Q(x)} e^{iax},\ a \geqslant 0 \right)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{+i}(f) = 2\pi i \lim_{z \to i} \big((z-i) \cdot f(z)\big)$$

$$= 2\pi i \cdot \frac{e^{a}}{2i}$$

$$= \pi \cdot e^{-|a|}$$

So, for $a \leq 0$: $\mathcal{F}\left[\frac{1}{1+x^2}\right](a) = \frac{1}{\sqrt{2\pi}} \cdot \pi \cdot e^{-|a|}$

$$= \sqrt{\frac{\pi}{2}}\, e^{-|a|}$$

For $a \geq 0$: The same computation (but with the semi-circle now in the lower half plane) gives the same formula.

The Laplace transform:

The Fourier transform: Used for signals that are defined on the <u>whole</u> real line.

Laplace transform: For signals defined on $t \geq 0$.

Both: Transform differential equations into algebraic equations.

Let $f: [0, +\infty) \longrightarrow \mathbb{C}$ be a piecewise continuous function. The Laplace transform of f:

$$\mathcal{L}[f](z) = \int_0^{+\infty} f(t)\, e^{-zt}\, dt$$

(z : complex!)

Sometimes: We use capital letters to denote the Laplace transform, for instance:

$$F(z) = \mathcal{L}[f](z), \quad G(z) = \mathcal{L}[g](z)$$

When is the integral well-defined (i.e. "converges") ?

We need $\displaystyle\int_0^{+\infty} |f(t)|\, |e^{-zt}|\, dt < +\infty$.

Suppose that, for some $\gamma_0 \in \mathbb{R}$, we have

$$\int_0^{+\infty} |f(t)|\, e^{-\gamma_0 t}\, dt < +\infty$$

(naively, this condition says that, as $t \to +\infty$, $|f(t)| < e^{\gamma_0 t}$ ...)

Then:

$$|L[f](z)| \leq \int_0^{+\infty} |f(t)|\, |e^{-zt}|\, dt$$

$$= \int_0^{+\infty} |f(t)| \cdot |e^{-Re(z) \cdot t - Im(z) \cdot t \cdot i}| dt$$

Since $\left|e^{-\mathrm{Im}(z)\cdot t\cdot i}\right| = 1$

$$= \int_0^{+\infty} |f(t)|\, e^{-\mathrm{Re}(z)\cdot t}\, dt$$

$$\overset{\mathrm{Re}(z)\geqslant\gamma_0}{\leq} \int_0^{+\infty} |f(t)|\, e^{-\gamma_0 t}\, dt$$

$$< +\infty.$$

Therefore, if such a γ_0 exists:
$\mathcal{L}[f](z)$ is defined at least over

$$U := \{ z \in \mathbb{C} : \mathrm{Re}(z) > \gamma_0 \}.$$

Definition:
A γ_0 such that $\mathcal{L}[f](z)$ is well-defined (as above) for any z with $\mathrm{Re}(z) > \gamma_0$ is called an abscissa of convergence.

(abscissa: Another name for the x-coordinate)

Example: If $f(t) = 0$ for $t \geq M$,

then $$\int_0^{+\infty} |f(t)| e^{-\gamma_a t} dt = \int_0^M |f(t)| e^{-\gamma_a t} dt$$

$$< +\infty \quad \forall \gamma_a \in \mathbb{R}$$

So in that case, $\mathcal{L}[f](z)$ is defined for $z \in \mathbb{C}$.

Remark: The "faster" the rate of growth for f as $t \to +\infty$ (or the "slower" the rate of decay), the smaller the domain $U \subseteq \mathbb{C}$ where $\mathcal{L}[f](z)$ is well-defined.

If γ_0 is an abscissa of convergence,

and $U = \{ z : \mathrm{Re}(z) > \gamma_0 \}$

then $\mathcal{L}[f]$ is holomorphic on U.

Let $F(z) = \mathcal{L}[f](z)$. Then

$$\frac{d}{dz} F[z] = \frac{d}{dz} \int_0^{+\infty} f(t) e^{-zt} \, dt$$

$$= \int_0^{+\infty} f(t) (-t) e^{-zt} \, dt$$

$$= - \mathcal{L}[t \cdot f(t)](z).$$

So:

Theorem: If $F(z) = \mathcal{L}[f](z)$,

$$F'(z) = -\mathcal{L}[t \cdot f(t)].$$

We will also use the following lemma (proven in the exercises):

Lemma: If γ_0 is an abscissa of convergence for $\mathcal{L}[f(t)]$, it is also for $\mathcal{L}[t \cdot f(t)]$.

Example 1.

Let $f: [0, +\infty) \longrightarrow \mathbb{C}$ be $f(t) = 1$ (constant). Then, for any $\gamma_0 > 0$:

$$\int_0^{+\infty} |f(t)|\, e^{-\gamma_0 t}\, dt = \int_0^{+\infty} e^{-\gamma_0 t}\, dt$$

$$= \left[\frac{e^{-\gamma_0 t}}{-\gamma_0} \right]_{t=0}^{t=+\infty}$$

$$= \frac{1}{\gamma_0} < \infty$$

So any $\gamma_0 > 0$ is an abscissa of convergence $\Rightarrow \mathcal{L}[f](z)$ is defined (and holomorphic) for $Re(z) > 0$.

And:

$$\mathcal{L}[f](z) = \int_0^{+\infty} 1 \cdot e^{-zt}\, dt \overset{z \neq 0}{=}$$

$$= \left[\frac{e^{-zt}}{-z} \right]_{t=0}^{t=+\infty} =$$

$$= -\frac{1}{z}\left(\lim_{t\to+\infty}\left(e^{-zt}\right) - e^{-0}\right)$$

But: $\lim_{t\to+\infty}\left(e^{-zt}\right) = 0$ for $\mathrm{Re}(z)>0$,

since $$|e^{-zt}| = \underbrace{|e^{-\mathrm{Re}(z)t}|}_{\to 0 \text{ as } t\to+\infty} \cdot \underbrace{|e^{-\mathrm{Im}(z)t\cdot i}|}_{=1}$$

So: $$\boxed{\mathcal{L}[1](z) = \frac{1}{z}}$$

<u>Example 2:</u>

Let $f: [0,+\infty) \to \mathbb{C}$, $f(t) = t$.

Then $\frac{d}{dz}\mathcal{L}[1](z) = -\mathcal{L}[t](z)$

$= -\mathcal{L}[f](z)$

So $\mathcal{L}[f](z) = -\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2}$

(Also holomorphic for $Re(z) > 0$, since as we saw in the lemma, if γ_0 is an abscissa of convergence for $\mathcal{L}[f]$, the same is true for $\mathcal{L}[t \cdot f]$ and any $\gamma_0 > 0$ is an abscissa of convergence for $\mathcal{L}[1]$)

Repeating this construction:

Theorem: For any $n \in \mathbb{N}$,

$$\mathcal{L}[t^n](z) = \frac{n!}{z^{n+1}}$$

defined for $\operatorname{Re}(z) > 0$.